

## THE IMPORTANCE OF BEING INTEGRABLE: OUT OF THE PAPER, INTO THE LAB

MURRAY T. BATCHELOR

*Centre for Modern Physics, Chongqing University,  
Chongqing 400044, China*

*Australian National University, Canberra, ACT 0200, Australia*

*E-mail: Murray.Batchelor@anu.edu.au*

The scattering matrix ( $S$ -matrix), relating the initial and final states of a physical system undergoing a scattering process, is a fundamental object in quantum mechanics and quantum field theory. The study of factorized  $S$ -matrices, in which many-body scattering factorizes into a product of two-body terms to yield an integrable model, has long been considered the domain of mathematical physics. Many beautiful results have been obtained over several decades for integrable models of this kind, with far reaching implications in both mathematics and theoretical physics. The viewpoint that these were only toy models changed dramatically with brilliant experimental advances in realizing low-dimensional quantum many-body systems in the lab. These recent experiments involve both the traditional setting of condensed matter physics and the trapping and cooling of atoms in optical lattices to engineer and study quasi-one-dimensional systems. In some cases the quantum physics of one-dimensional systems is arguably more interesting than their three-dimensional counterparts, because the effect of interactions is more pronounced when atoms are confined to one dimension. This article provides a brief overview of these ongoing developments, which highlight the fundamental importance of integrability.

*Keywords:* non-diffracting scattering, Yang-Baxter integrability, Bethe Ansatz.

### 1. Introduction

The goal of theoretical physics is to develop theories for the physical description of reality. This provides ample enough motivation to study model systems which are constructed to capture the essential physics of a given problem. The strong predictive power of such basic models is one of the triumphs of theoretical physics. On the other hand, mathematical models of this kind often become interesting in their own right, leading into the realms of mathematical physics. If the mathematical structures are sufficiently rich then progress can be inspired in mathematics itself.

The particular models we have in mind here are the so-called integrable models of statistical mechanics and quantum field theory. Their origin dates back to soon after the development of quantum mechanics, when the eigenspectrum of the one-dimensional spin- $\frac{1}{2}$  Heisenberg chain was obtained in exact closed form by Hans Bethe.<sup>1</sup> The underlying Bethe Ansatz for the wave function is the hallmark of the

integrable models to be discussed here. Indeed, these models can be referred to as being Bethe Ansatz integrable. Some key examples from a golden period in the 1960's when the inner workings of the one-dimensional models were uncovered are:

- Bose gas, Lieb & Liniger,<sup>2</sup> McGuire,<sup>3</sup> Berezin *et al.*<sup>4</sup> (1963,1964)
- Fermi gas,  $M = 1$ , McGuire<sup>5</sup> (1965)
- Fermi gas,  $M = 2$ , Flicker & Lieb<sup>6</sup> (1967)
- Fermi gas,  $M$  arbitrary, Gaudin,<sup>7,8</sup> Yang<sup>9,10</sup> (1967,1968)
- Fermi gas, higher spin, Sutherland<sup>11</sup> (1968)
- Hubbard model, Lieb & Wu<sup>12</sup> (1968)

Here  $M$  is the number of spins flipped from the ferromagnetic state.

Another strand of developments during this golden period – later seen to be not unrelated – was sparked by Lieb's exact solution of the ice-type models, which culminated in Baxter's invention of the commuting transfer matrix and functional equation method to solve the eight-vertex model.<sup>13</sup>

There is a deep reason for why these models are integrable. At the heart is the Yang-Baxter relation, which has appeared in many guises.<sup>a</sup> Our interest here is in the context of quantum many-body systems for which the key ingredient is the scattering matrix ( $S$ -matrix). In particular, our interest is in models for which the  $S$ -matrix of an  $N$ -particle system factorizes into a product of  $N(N-1)/2$  two-body  $S$ -matrices. For models confined to one space dimension this factorisation is represented as a space-time scattering diagram in Fig. 1.

The condition of integrability is equivalent to a condition of no diffraction.<sup>3,20</sup> Indeed, the Bethe Ansatz can only be applied when there is non-diffracting scattering. The notions of non-diffracting scattering and quantum integrability are essentially equivalent in the present context. This is one of the best available definitions of quantum integrability.<sup>24</sup>

In the next section the above examples will be discussed briefly.

## 2. Factorized scattering and key integrable models

In a completely integrable system the three-body  $S$ -matrices corresponding to the two diagrams in Fig. 1(b) are equal and have the factorisation equation

$$S(1, 2, 3) = S(2, 3)S(1, 3)S(1, 2) = S(1, 2)S(1, 3)S(2, 3) \quad (1)$$

where  $S(i, j)$  is the two-body  $S$ -matrix acting on states  $i$  and  $j$ . In the Yang-Baxter language, this form of  $S(i, j)$  is identical to Yang's operator  $X_{ij} = P_{ij}Y_{ij}^{ij}$ , or equivalently, to Baxter's  $R$ -matrices.<sup>b</sup>

<sup>a</sup>See, e.g., the various books, review articles and lecture notes in Refs. 13–22, which is by no means a complete list. Indeed, the brief overview given in the present article is necessarily incomplete. In some sense this article is a sequel to Ref. 23.

<sup>b</sup>Yang's masterstroke was to translate McGuire's geometric construction into operator form.<sup>9,10</sup>

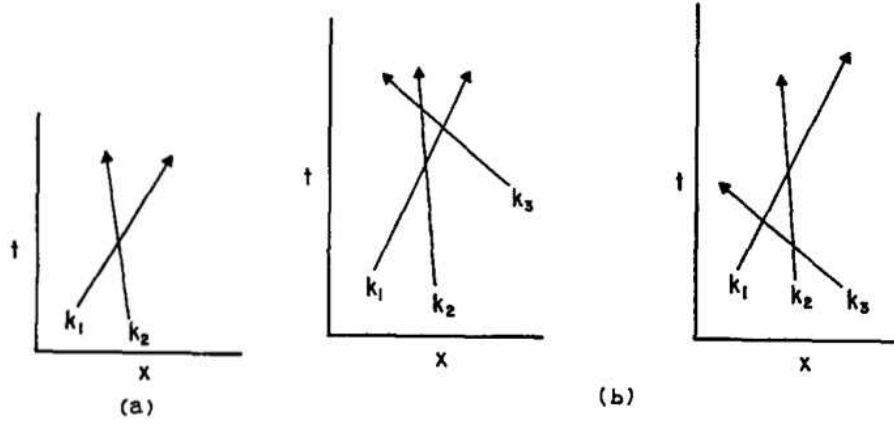


Fig. 1. Space-time plots for (a) two- and (b) three-particle problems. The point is that the two factorisations in (b) are equivalent. From McGuire.<sup>3</sup>

### 2.1. bosons

The hamiltonian of  $N$  interacting spinless bosons on a line of length  $L$  ( $\hbar = 2m = 1$ ) with point interactions is

$$\mathcal{H} = - \sum_{i=1}^N \frac{\partial^2}{\partial x_i^2} + 2c \sum_{1 \leq i < j \leq N} \delta(x_i - x_j) \quad (2)$$

where  $x_i$  are the boson co-ordinates and  $c$  is the interaction strength. This is the model solved in Lieb and Liniger<sup>2</sup> by means of the Bethe Ansatz wavefunction<sup>c</sup>

$$\psi(x_1, \dots, x_N) = \sum_P A(P) \exp(i \sum_{j=1}^N k_{Pj} x_j) \quad (3)$$

which gives the energy eigenvalues

$$\mathcal{E} = \sum_{j=1}^N k_j^2 \quad (4)$$

in terms of the roots  $k_j$  of Bethe equations of the form

$$\exp(ik_j L) = - \prod_{\ell=1}^N \frac{k_j - k_\ell + ic}{k_j - k_\ell - ic} \quad \text{for } j = 1, \dots, N. \quad (5)$$

This is arguably the simplest set of known Bethe equations, for which all of the roots  $k_j$  are real in the repulsive regime  $c > 0$ .

<sup>c</sup>The amplitudes  $A(P)$  involve a sum over permutations  $P = (P_1, \dots, P_N)$  of  $(1, \dots, N)$ .

For this model the two-body  $S$ -matrix element for  $k_1 < k_2$  is  $S(k_2, k_1) = S(k_2 - k_1) = S(p)$ , where

$$S(p) = \frac{p - ic}{p + ic} \quad (6)$$

with  $p$  the rapidity and  $S(p)S(-p) = 1$ .

In the analysis of this model it is convenient to define the dimensionless interaction parameter  $\gamma = c/n$  in terms of the number density  $n = N/L$ . A cartoon of the atom distributions, representing the ‘fermionisation’ of the one-dimensional interacting Bose gas with increasing  $\gamma$  is shown in Fig. 2.

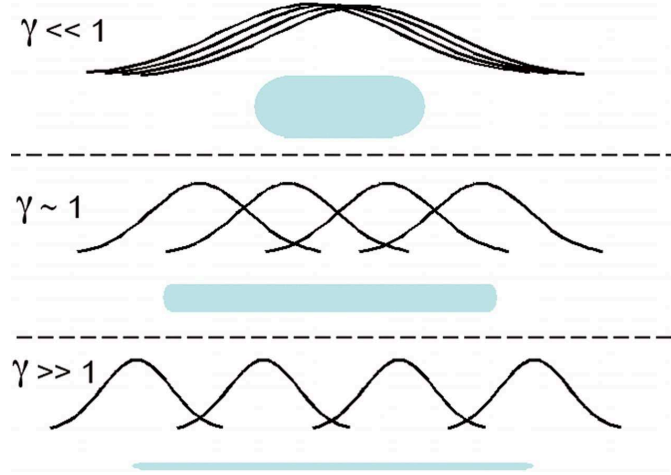


Fig. 2. Cartoon showing the ‘fermionisation’ of bosons as the interaction strength  $\gamma$  is increased. For  $\gamma \ll 1$  the behaviour is like a condensate, whereas for  $\gamma \gg 1$  the behaviour is like the hard-core bosons of the Tonks-Girardeau gas. From Kinoshita, Wenger and Weiss.<sup>25</sup>

The attractive regime  $c < 0$  has also been of interest. Inspired by Monte Carlo results which predicted the existence of a super Tonks-Girardeau gas-like state in the attractive interaction regime of quasi-one-dimensional Bose gases,<sup>26</sup> it was shown that a super Tonks-Girardeau gas-like state corresponds to a highly-excited Bethe state in the integrable Bose gas with attractive interactions, for which the bosons acquire hard-core behaviour.<sup>27</sup> The large kinetic energy inherited from the Tonks-Girardeau gas – as the interaction is switched from strongly repulsive to strongly attractive – in a Fermi-pressure-like manner, prevents the gas from collapsing.

## 2.2. *fermions*

The hamiltonian of the one-dimensional fermion problem is similar to Eq. (2), with

$$\mathcal{H} = - \sum \frac{\partial^2}{\partial x_i^2} - \sum \frac{\partial^2}{\partial y_i^2} + 2c \sum \delta(x_i - y_j) \quad (7)$$

where  $x_i$  and  $y_i$  are the co-ordinates of the spin-up and spin down fermions. In this case there are a total of  $N$  interacting two-component fermions on a line of length  $L$ , with  $M$  the number of spin-down fermions. The energy expression is also the same as Eq. (4). However, because spin is involved, the Bethe equations are now of the more complicated – nested – form<sup>7–9</sup>

$$\exp(ik_j L) = \prod_{\ell=1}^M \frac{k_j - \Lambda_\ell + \frac{1}{2}ic}{k_j - \Lambda_\ell - \frac{1}{2}ic} \quad (8)$$

$$\prod_{\ell=1}^N \frac{\Lambda_\alpha - k_\ell + \frac{1}{2}ic}{\Lambda_\alpha - k_\ell - \frac{1}{2}ic} = - \prod_{\beta=1}^M \frac{\Lambda_\alpha - \Lambda_\beta + ic}{\Lambda_\alpha - \Lambda_\beta - ic} \quad (9)$$

for  $j = 1, \dots, N$  and  $\alpha = 1, \dots, M$ . We may define the polarization by  $P = (N - 2M)/N$ . The special case  $M = N/2$  for which  $P = 0$  is known as the balanced case.

The matrix form of the associated wavefunction and  $S$ -matrix are not discussed here, rather the reader is referred to the literature.<sup>14,20,28</sup>

In the attractive regime the Bethe roots tend to form pairs which can be broken by applying a magnetic field to the hamiltonian Eq. (7). The quantum critical points distinguishing the different quantum phases (see Fig. 3) can be calculated analytically and the full phase diagram mapped out (see Fig. 4).<sup>28</sup>



Fig. 3. This figure is essentially the zero temperature phase diagram of the Gaudin-Yang model as a function of magnetic field  $h$  for given chemical potential. The three phases are the fully paired (BCS) phase, which is a quasi-condensate with zero polarization ( $P = 0$ ), the fully polarized (Normal) phase with  $P = 1$ , and the partially polarized (FFLO) phase where  $0 < P < 1$ . The FFLO phase can be viewed as a mixture of pairs and leftover (unpaired) fermions. For given chemical potential, the FFLO phase is separated from the BCS phase and the normal phase by the quantum critical points  $h_{c1}$  and  $h_{c2}$ . From Zhao and Liu.<sup>29</sup>

Much has been written about the one-dimensional Hubbard model,<sup>12</sup> which is also of fundamental importance.<sup>31</sup> The model describes interacting electrons in narrow energy bands and in the continuum limit is equivalent to the interacting two-component fermion model.

### 2.3. $E_8$ and the Ising model

Like the result of the alluring call of the Sirens in Greek mythology, attempts to solve the two-dimensional classical Ising model in a magnetic field – and its one-dimensional quantum counterpart – have foundered on the rocks. Fortunately

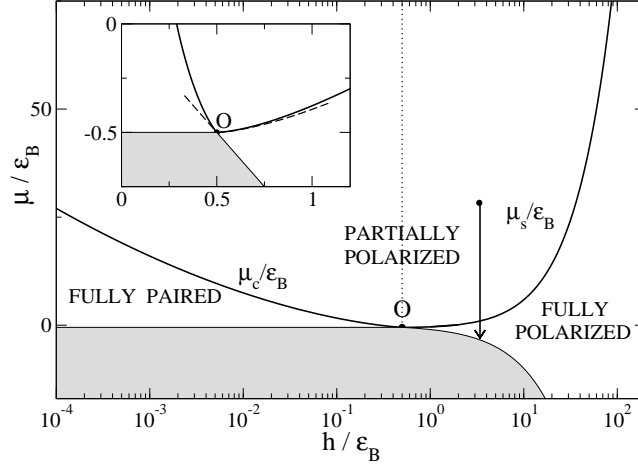


Fig. 4. Phase diagram of the Gaudin-Yang model as a function of chemical potential and magnetic field obtained from using numerical solution of the Bethe equations. From Orso.<sup>30</sup>

there is a modern day Orpheus. Zamolodchikov<sup>32</sup> discovered a remarkable integrable quantum field theory containing eight massive particles with a reflectionless factorized  $S$ -matrix. This is the  $c = \frac{1}{2}$  conformal field theory (corresponding to the critical Ising model) perturbed with the spin operator  $\phi_{1,2} = \phi_{2,2}$  of dimension  $(\frac{1}{16}, \frac{1}{16})$ . Up to normalisation, the masses  $m_i$  of these particles coincide with the components  $S_i$  of the Perron-Frobenius vector of the Cartan matrix of the Lie algebra  $E_8$ :  $m_i/m_j = S_i/S_j$ .

With normalisation  $m_1 = 1$ , the masses<sup>d</sup> are<sup>32</sup>

$$\begin{aligned}
 m_2 &= 2 \cos \frac{\pi}{5} &= 1.618\,033\dots \\
 m_3 &= 2 \cos \frac{\pi}{30} &= 1.989\,043\dots \\
 m_4 &= 4 \cos \frac{\pi}{5} \cos \frac{7\pi}{30} &= 2.404\,867\dots \\
 m_5 &= 4 \cos \frac{\pi}{5} \cos \frac{2\pi}{15} &= 2.956\,295\dots \\
 m_6 &= 4 \cos \frac{\pi}{5} \cos \frac{\pi}{30} &= 3.218\,340\dots \\
 m_7 &= 8 \cos^2 \frac{\pi}{5} \cos \frac{7\pi}{30} &= 3.891\,156\dots \\
 m_8 &= 8 \cos^2 \frac{\pi}{5} \cos \frac{2\pi}{15} &= 4.783\,386\dots
 \end{aligned}$$

The  $S$ -matrix of this model is particularly impressive. The  $S$ -matrix element describing the scattering of the lightest particles is given by<sup>32</sup>

$$S_{1,1}(\beta) = \frac{\tanh\left(\frac{\beta}{2} + i\frac{\pi}{6}\right) \tanh\left(\frac{\beta}{2} + i\frac{\pi}{5}\right) \tanh\left(\frac{\beta}{2} + i\frac{\pi}{30}\right)}{\tanh\left(\frac{\beta}{2} - i\frac{\pi}{6}\right) \tanh\left(\frac{\beta}{2} - i\frac{\pi}{5}\right) \tanh\left(\frac{\beta}{2} - i\frac{\pi}{30}\right)} \quad (10)$$

<sup>d</sup>Note the appearance of the golden ratio for  $m_2/m_1$ .

where  $\beta$  is the rapidity. The other elements are uniquely determined by the bootstrap program.

The  $E_8$  theory is conjectured to describe the scaling limit of the two-dimensional classical Ising model in a magnetic field.<sup>e</sup> The first several masses were soon confirmed numerically for the one-dimensional quantum counterpart, the quantum Ising chain with transverse and longitudinal fields.<sup>34</sup> A realisation exists in terms of the dilute  $A_3$  lattice model<sup>35</sup> – an exactly solved lattice model in the same universality class as the two-dimensional Ising model in a magnetic field – from which the  $E_8$  mass spectrum has been derived.<sup>36,37</sup>

As we shall see further below, the fact that the emergence of such an exotic symmetry as  $E_8$  can be observed in the lab is quite remarkable.

### 3. Experiments

In this section a brief sketch is given of experiments which have made contact with the models discussed above.<sup>f</sup> These experiments have been a result of the ongoing ‘virtuoso triumphs’ in experimental techniques – in cold atom optics and in the more traditional setting of condensed matter physics. As a result it is now possible to probe and understand the physics of key quantum many-body systems which should ultimately be of benefit to quantum technology.

#### 3.1. bosons

Experiments on the trapping and cooling of bosonic atoms in tight one-dimensional waveguides and related theoretical progress have been recently reviewed.<sup>38,39</sup> Most importantly, it is possible to confine atoms to effectively one-dimensional tubes and to vary the interaction strength between atoms, both in the repulsive and attractive regimes.

One of the early experiments which made contact with the one-dimensional Lieb-Liniger model of interacting bosons measured local pair correlations in bosonic Rb atoms by photoassociation. The local pair correlation function  $g^{(2)}$  is proportional to the probability of observing two particles in the same location. The experimental measurement of  $g^{(2)}$  by Kinoshita, Wenger and Weiss<sup>40</sup> is shown in Fig. 5. As expected, the curve drops off towards zero as the interaction strength increases, just like in a non-interacting Fermi gas (recall Fig. 2).

Experiments have also been performed on one-dimensional bosons in the attractive regime. In particular, using a tunable quantum gas of bosonic cesium atoms, Haller *et al.*<sup>41</sup> realized and controlled in one-dimensional geometry a highly excited quantum phase – the super Tonks-Girardeau gas – that is stabilized in the presence of attractive interactions by maintaining and strengthening quantum correlations across a confinement-induced resonance (see Fig. 6). They diagnosed the crossover

<sup>e</sup>For recent work with regard to scaling and universality, see Ref. 33 and references therein.

<sup>f</sup>The particular experiments chosen are selective and by no means exhaustive.

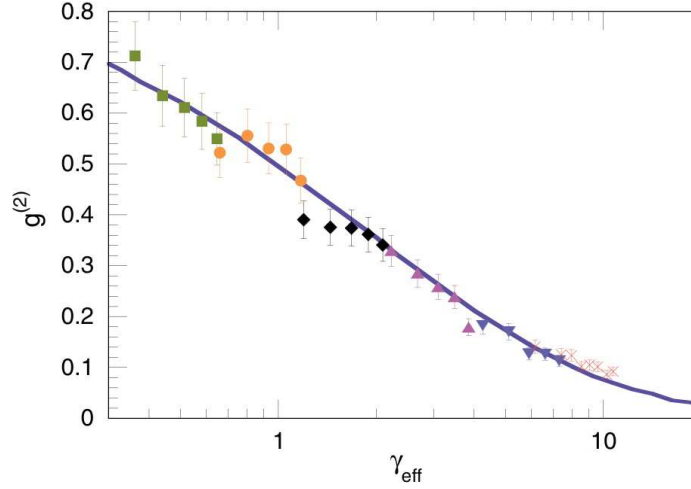


Fig. 5. Local pair correlation function  $g^{(2)}$  obtained from photoassociation rates vs effective interaction strength. The solid line is obtained from the Lieb-Liniger model. From Kinoshita, Wenger and Weiss.<sup>40</sup>

from repulsive to attractive interactions in terms of the stiffness and energy of the system. This opened up the experimental study of metastable, excited, many-body phases with strong correlations.

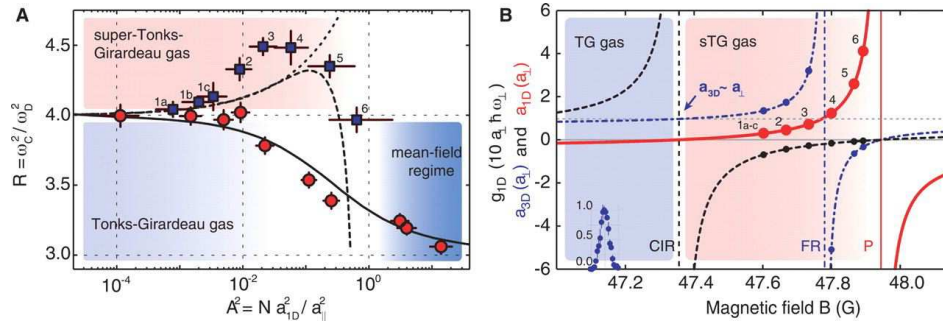


Fig. 6. Plots of experimental data providing evidence for the super Tonks-Girardeau gas. From Haller *et al.*<sup>41</sup>

### 3.2. fermions

Theoretical progress and experiments on fermionic atoms confined to one-dimension have been recently reviewed.<sup>28</sup> Of particular relevance here is the experiment performed at Rice University using fermionic  $^6\text{Li}$  atoms.<sup>42</sup> The system has attractive



interactions with a spin population imbalance caused by a difference in the number of spin-up and spin-down atoms. Experimentally, the gas is dilute and strongly interacting. The key features of the phase diagram (recall Fig. 4) have been experimentally confirmed using finite temperature density profiles (see Fig. 7).<sup>42</sup> The system has a partially polarized core surrounded by either fully paired or fully polarized wings at low temperatures, in agreement with theoretical predictions.<sup>28</sup> More generally, this work experimentally verifies the coexistence of pairing and polarization at quantum criticality.

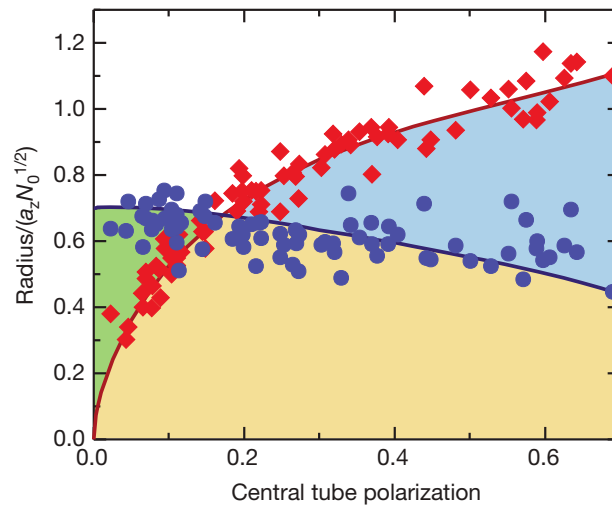


Fig. 7. Experimental phase diagram of one-dimensional two-component fermions as a function of polarization. The red diamonds and blue circles denote the scaled radii of the axial density difference and the minority state axial density, respectively. The solid lines follow from the Gaudin-Yang model. From Liao *et al.*<sup>42</sup>

In further developments, experiments have also been performed with just two distinguishable  $^6\text{Li}$  atoms.<sup>43</sup> This provides an experimental study of one-dimensional fermionisation as a function of the interaction strength. For a magnetic field below the confined induced resonance two interacting fermions form a Tonks-Girardeau state whereas a super Tonks-Girardeau gas is created when the magnetic field is above the resonance value. Quasi one-dimensional systems consisting of up to six ultracold fermionic atoms in two different spin states with attractive interactions have also been studied experimentally,<sup>44</sup> including the crossover from few to many-body physics.<sup>45</sup>

### 3.3. $E_8$ and the quasi-1D Ising ferromagnet $\text{CoNb}_2\text{O}_6$

In an experiment in the traditional setting of condensed matter physics, Coldea *et al.*<sup>46</sup> realised a quasi-one-dimensional Ising ferromagnet in  $\text{CoNb}_2\text{O}_6$  (cobalt niobate) tuned through its quantum critical point using strong transverse magnetic fields. The underlying Ising hamiltonian

$$H = -J \sum_i s_i^z s_{i+1}^z - h \sum_i s_i^x - h_z \sum_i s_i^z \quad (11)$$

has a quantum critical point at  $h = h_c = J/2$  for  $h_z = 0$ . In the scaling limit sufficiently close to the quantum critical point, i.e.,  $h_z \ll J, h = h_c$ , the spectrum is predicted to be described by Zamolodchikov's  $E_8$  mass spectrum. Coldea *et al.*<sup>46</sup> were able to observe the spectrum by neutron scattering. In particular, they were able to observe evidence for the first few  $E_8$  masses, see Fig. 8.

In fact the integrable theory provides many more exact predictions than experiments have been able to test so far,<sup>47</sup> involving, for example, correlation functions.<sup>48</sup> Since the work of Coldea *et al.*<sup>46</sup> it is reasonable to expect further progress on the experimental side.

## 4. Concluding remarks

We have seen through the few examples given here that the twin concepts of non-diffractive and factorized scattering – embodied in the Yang-Baxter equation – have captured the fundamental physics of some key interacting quantum many-body systems. It could hardly have been imagined in the 1960's that such mathematical models would some day make contact with experiment. The philosophy and pioneering spirit of the 1960's was captured at that time in the compilation of introductory material and original articles<sup>§</sup> in the book *Mathematical Physics in One Dimension* by Lieb and Mattis.<sup>49</sup> Over the following decades further theoretical progress and the striking developments in experimental technology have revealed that physics in one dimension is indeed a particularly rich and worthwhile pursuit,<sup>50</sup> and *does* provide a path to understanding nature.

There has long been a school of thought, with which Professor Dyson concurs, that mathematical models should only be tackled in earnest if there is a prospect that some day they may be relevant to experiments. Yet we have seen from the examples of the one-dimensional Bose and Fermi gases that it may take up to, and even more than, 40 years before mathematical models of this kind move out of

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<sup>§</sup> Including material and a reprinted paper by F. J. Dyson on the dynamics of disordered chains. On a personal note, we are all familiar with Dyson's early mastery of the  $S$ -matrix in quantum electrodynamics and his brilliant synthesis of the different formulations of quantum electrodynamics due to Tomonaga, Schwinger and Feynman. My favourite Dyson moment is his legendary common room encounter with Hugh Montgomery in the early 1970's which set alight the discovery of the remarkable and deep connection between the distribution of zeros of the Riemann zeta function and random Hermitian matrices. Dyson's perspective, of course, was from his earlier seminal work on the statistical features of the level spacings of quantum systems.

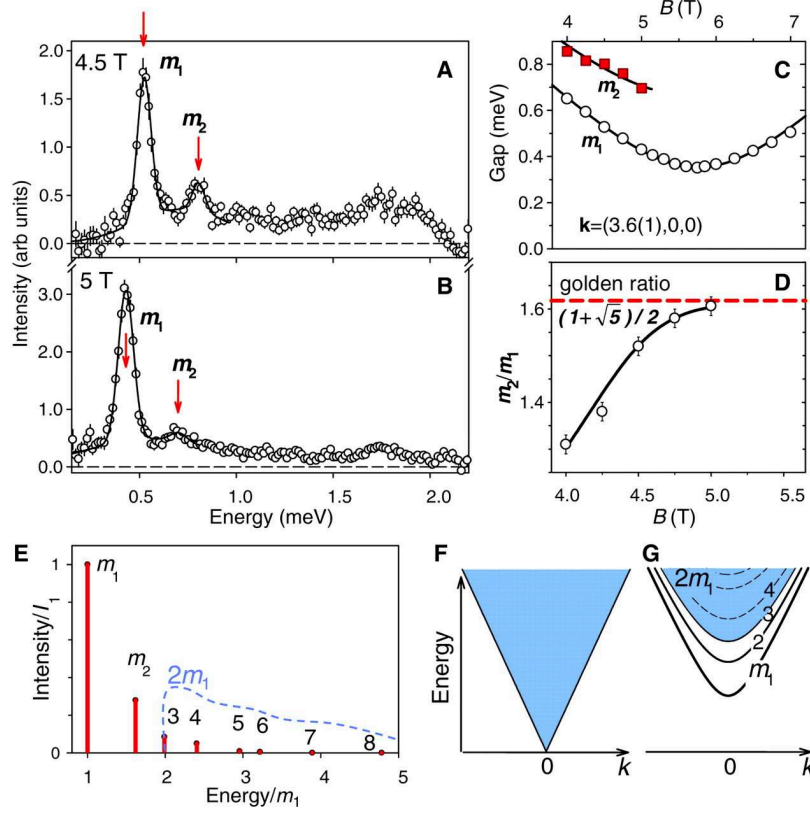


Fig. 8. Various plots showing experimental evidence for the first few masses of the  $E_8$  mass spectrum in the transverse Ising chain. From Coldea *et al.*<sup>46</sup>

the paper and into the lab. Fortunately this has been during the lifetime of those involved in the pioneering developments of the 1960's and later.

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